Relations

A **relation** R (or $\sim$) on a set S is a subset $R \subseteq S \times S$.

Notation: If $(a,b) \in R$ then we write aRb. (a is related to b)

Exs:

1) $S = \{1, 2, 3\}$

$\left(S \times S = \{(1,1), (1,2), \ldots, (3,3)\} \quad |S \times S| = 9 \right)$

$R_1 = \{(1,2), (2,3)\}$

$R_2 = \{(1,1), (2,2), (3,3)\}$

$R_3 = \{(1,1), (1,3), (3,1), (3,3)\}$

$R_4 = S \times S$

2) If $f: X \to X$ is a function then we can define a relation R on X by

$$R = \{(x, f(x)) : x \in X\}. \quad (\text{"graph" of } f: X \to X)$$

3) If $f: X \to Y$ is a function, we can define a relation R on X by

$$R = \{(x, x') : f(x) = f(x')\}. \quad (\underline{\text{fibers}} \text{ of } f: X \to Y)$$

4) Let $\mathcal{T}$ be the set of all triangles in $\mathbb{R}^2$ and, for triangles $T_1, T_2 \in \mathcal{T}$, write $T_1 \sim T_2$ iff T_1 and T_2 are similar triangles.

Then

$$\{(T, T') \in \mathcal{T} \times \mathcal{T} : T \sim T'\}$$

is a relation on $\mathcal{T}$.

5) Define a relation R on $\mathbb{R}$ by

$$R = \{(x, y) \in \mathbb{R}^2 : x \leq y\}.$$

6) Let $n \in \mathbb{N}$ and for $a, b \in \mathbb{Z}$ write

$$a \sim b \text{ iff } n \mid a - b.$$

Then $\{(a, a') \in \mathbb{Z}^2 : a \sim a'\}$ is a relation on $\mathbb{Z}$.

Ex: $n = 2$,

$$\{(0,0), (0,-2), (0,2), (0,-4), (0,4), \ldots,$$
$$(1,1), (1,-1), (1,3), (1,-3), (1,5), \ldots,$$
$$(2,2), (2,0), (2,4), \ldots$$
$$\vdots \quad \ldots\} = \{(a, a') : 2 \mid a - a'\}$$

Some properties that a relation R on S may satisfy:

- R is <u>reflexive</u> iff, $\forall x \in S$, $(x,x) \in R$.
- R is <u>symmetric</u> iff, $\forall x, y \in S$,
  if $(x,y) \in R$ then $(y,x) \in R$.
- R is <u>transitive</u> iff, $\forall x, y, z \in S$,
  if $(x,y) \in R$ and $(y,z) \in R$ then $(x,z) \in R$.

If R is reflexive, symmetric, and transitive, then we say that it is an <u>equivalence relation</u>.

Exs. from before:

1) $S = \{1, 2, 3\}$

$R_1 = \{(1,2), (2,3)\}$ (not an equiv. rel.)

- not reflexive: $(1,1) \notin R_1$.
- not symmetric: $(1,2) \in R_1$ but $(2,1) \notin R_1$.
- not transitive: $(\underset{x}{1}, \underset{y}{2}), (2, \underset{z}{3}) \in R_1$ but $(1,3) \notin R_1$.

$S = \{1, 2, 3\}$

$R_2 = \{(1,1), (2,2), (3,3)\}$ (equivalence relation)

- reflexive ✓
- symmetric ✓
- transitive ✓

If $(x,y), (y,z) \in R_2$ then $x = y$ and $y = z$,

so $x = z$ and $(x,z) \in R_2$.

$S = \{1, 2, 3\}$

$R_3 = \{(1,1), (1,3), (3,1), (3,3)\}$ (not an equiv. rel.)

- not reflexive: $(2,2) \notin R_3$
- symmetric ✓
- transitive: ✓

If $(x,y), (y,z) \in R_3$ then $x = 1$ or 3 and

$z = 1$ or $3 \Rightarrow (x,z) \in R_3$.

(true for any set S)

$R_4 = S \times S$ (equivalence relation)

- reflexive ✓
- symmetric ✓
- transitive ✓

2) $f: X \to X$

$R = \{(x, f(x)) : x \in X\}$ (eq. rel. iff $f(x) = x$, $\forall x \in X$).

- will be reflexive iff $f(x) = x$, $\forall x \in X$.
- may or may not be symmetric
- may or may not be transitive

but will be both if $f(x) = x$, $\forall x \in X$.

3) $f: X \to Y$ (fibers of $f: X \to Y$)

$R = \{(x, x') : f(x) = f(x')\}$ (equivalence relation)

- reflexive: ✓

  $\forall x \in X$, $f(x) = f(x) \Rightarrow (x, x) \in R$.

- symmetric: ✓

  If $(x, y) \in R$ then

  $f(x) = f(y) \Rightarrow f(y) = f(x) \Rightarrow (y, x) \in R$.

- transitive: ✓

  If $(x, y), (y, z) \in R$ then

  $f(x) = f(y) = f(z) \Rightarrow (x, z) \in R$.

4) $T_1 \sim T_2$ iff T_1 and T_2 are similar triangles

(equivalence relation)

5) $S=\mathbb{R}$, $xRy \Leftrightarrow x \le y$ (not an equiv. rel.)

- reflexive: ✓

  $\forall x \in \mathbb{R},\ x \le x$

- not symmetric: $1 \le 2$ but $2 \not\le 1$.

- transitive: ✓

  If $x \le y$ and $y \le z$ then $x \le z$.

6) Let $n \in \mathbb{N}$ and for $a, b \in \mathbb{Z}$ write

$a \sim b$ iff $n \mid a-b$. (equivalence relation)

- reflexive: ✓

  $(0 = n \cdot 0)$

  $\forall a \in \mathbb{Z},\ a-a=0 \Rightarrow n \mid a-a \Rightarrow a \sim a.$

- symmetric: ✓

  $(a-b=nk \Rightarrow b-a=n(-k))$

  If $a \sim b$ then $n \mid a-b \Rightarrow n \mid b-a \Rightarrow b \sim a$

- transitive: ✓

  Suppose $a \sim b$ and $b \sim c$. Write $a-b=nk$ and $b-c=n\ell$. Then

  $$a-c=(a-b)+(b-c)=nk-n\ell=n(k-\ell)$$

  $$\Rightarrow n \mid a-c \Rightarrow a \sim c.$$

Notation: For $a, b \in \mathbb{Z}$, $a \equiv b \bmod n \Leftrightarrow n \mid a-b$.

(a equals b modulo n)

If $\sim$ is an equivalence relation on S then, $\forall x \in S$, the <u>equivalence class of x</u> is the subset of S defined by

$$\bar{x} = \{y \in S : x \sim y\}.$$

Every element of this equivalence class is a <u>representative</u> for the class.

Facts: 1) $\forall x \in S$, $x \in \bar{x}$. ($\sim$ is refl.)

2) $\forall y \in \bar{x}$, $\bar{y} = \bar{x}$.

Pf: Suppose $y \in \bar{x}$.

($x \sim z$)

- If $z \in \bar{x}$ then, since

($\sim$ is sym.)

$$y \in \bar{x} \Rightarrow x \sim y \Rightarrow y \sim x,$$

we have that

($\sim$ is trans.)

$$y \sim x \text{ and } x \sim z \Rightarrow y \sim z \Rightarrow z \in \bar{y}.$$

Therefore, $\bar{x} \subseteq \bar{y}$.

- Since $x \sim y \Rightarrow y \sim x \Rightarrow x \in \bar{y}$, by reversing the roles of x and y in the above argument we have that $\bar{y} \subseteq \bar{x}$.

Conclusion: $\bar{x} = \bar{y}$. ∎

3) If $\overline{x_1} \cap \overline{x_2} \neq \phi$ then $\overline{x_1} = \overline{x_2}$.

Pf: If $\exists\, y \in \overline{x_1} \cap \overline{x_2}$ then, by the previous fact, $\overline{y} = \overline{x_1}$ and $\overline{y} = \overline{x_2}$, so $\overline{x_1} = \overline{x_2}$. ∎

Exs from before:

1) $S = \{1, 2, 3\}$

$R_2 = \{(1,1), (2,2), (3,3)\}$ (equivalence relation)

$\overline{1} = \{1\}$ $\quad \overline{2} = \{2\}$ $\quad \overline{3} = \{3\}$

$R_4 = S \times S$ (equivalence relation)

$\forall x \in S$, $\overline{x} = \{y \in S : (x,y) \in R_4\} = S$.

(only one equiv. class)

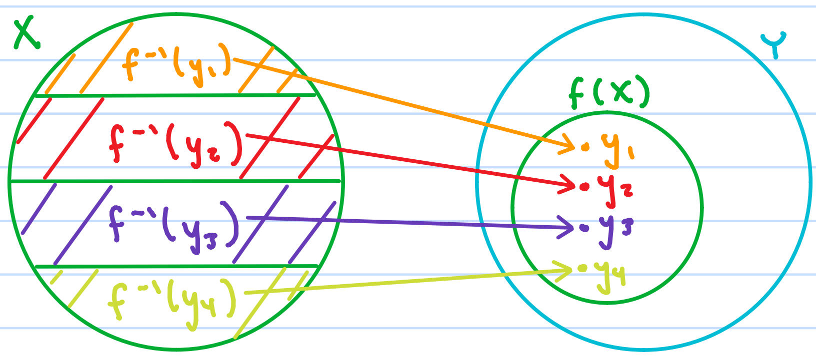
3) $f: X \to Y$ (*fibers* of $f: X \to Y$)

$R = \{(x, x'): f(x) = f(x')\}$ (equivalence relation)

$\forall y \in f(X)$, $\exists x \in X$ s.t. $f(x) = y$.

Let $f^{-1}(y) = \bar{x}$. (Note: f may not be bijective)

preimage of y or fiber over y



Then:

- $X = \bigcup_{y \in f(X)} f^{-1}(y)$.

  Pf: $\forall x \in X$, $x \in f^{-1}(f(x))$. ∎

- If $y, y' \in f(X)$, $y \neq y'$, then $f^{-1}(y) \cap f^{-1}(y') = \emptyset$.

  Pf: If $x \in f^{-1}(y)$ and $x' \in f^{-1}(y')$ then

  $f(x) = y$ and $f(x') = y'$, but $y \neq y'$,

  so $(x, x') \notin R \Rightarrow x' \notin f^{-1}(x)$

  and $(x', x) \notin R \Rightarrow x \notin f^{-1}(x')$. ∎

6) Let $n \in \mathbb{N}$ and for $a, b \in \mathbb{Z}$ write

$a \sim b$ iff $n \mid a-b$. (equivalence relation)

($a \equiv b \mod n$)

Equivalence classes:

$\overline{0} = \{\dots, -2n, -n, 0, n, 2n, \dots\}$

$\overline{1} = \{\dots, -2n+1, -n+1, 1, n+1, 2n+1, \dots\}$

$\overline{2} = \{\dots, -2n+2, -n+2, 2, n+2, 2n+2, \dots\}$

$\vdots$

$\overline{n-1} = \{\dots, -n-1, -1, n-1, 2n-1, 3n-1, \dots\}$

residue classes modulo n

- $\mathbb{Z} = \bigcup_{r=0}^{n-1} \overline{r}$.

Pf: By the Division algorithm, $\forall b \in \mathbb{Z}$, $\exists\, q, r \in \mathbb{Z}$ s.t. $b = qn + r$, $0 \le r \le n-1$.

Then $b - r = qn \Rightarrow b \equiv r \mod n$

$\Rightarrow b \in \overline{r}$. $\blacksquare$

- The residue classes $\overline{0}, \overline{1}, \dots, \overline{n-1}$ are disjoint.

Pf: Let $0 \le r \le r' \le n-1$ suppose that $b \in \mathbb{Z}$, and that $b \in \overline{r} \cap \overline{r'}$. Then, by fact 3 from before, $\overline{r} = \overline{r'}$. Then $r \in \overline{r'} \Rightarrow n \mid r' - r$.

But $0 \le r' - r \le n-1 \Rightarrow r' = r$. $\blacksquare$

<u>Equivalence relations and partitions</u>

(indexing set)

Let S be a set and $\forall i \in I$ let $A_i \subseteq S$. We say that $\{A_i\}_{i \in I}$ is a <u>partition</u> of S if:

i) $\forall i \in I$, $A_i \neq \emptyset$,

ii) $S = \bigcup_{i \in I} A_i$, and

$(\forall i, j \in I \text{ with } i \neq j,\ A_i \cap A_j = \emptyset)$

iii) $\{A_i\}_{i \in I}$ is pairwise disjoint.

- Every partition determines an equivalence relation:

  Suppose $\{A_i\}_{i\in I}$ is a partition of S.

  Define $\sim$ on S by

  $$x \sim y \iff \exists i \in I \text{ s.t. } x, y \in A_i.$$

  Then $\sim$ is: (equivalence relation)

  - reflexive: ✓

    $\forall x \in S$, $\exists i \in I$ s.t. $x \in A_i$. (property ii)

    Therefore $x \sim x$.

  - symmetric ✓
  - transitive: ✓

    If $x \sim y$ then $\exists i \in I$ s.t. $x, y \in A_i$.

    If $y \sim z$ then $\exists j \in I$ s.t. $y, z \in A_j$.

    But $y \in A_i \cap A_j \Rightarrow i = j$ (property iii)

    $\Rightarrow x, z \in A_i \Rightarrow x \sim z$.

- Every equivalence relation determines a partition:

Suppose $\sim$ is an equiv. rel. on S, and let $\{A_i\}_{i\in I}$ be the collection of distinct equivalence classes of $\sim$. Then:

i) $\forall i \in I,\ A_i \neq \emptyset$: ✓

$\forall i \in I,\ \exists x \in S$ s.t. $A_i = \overline{x}$.

But $x \in \overline{x} \Rightarrow A_i \neq \emptyset$.

(fact 1)

ii) $S = \bigcup_{i\in I} A_i$: ✓

$\forall x \in S,\ x \in \overline{x}$, and $\overline{x} = A_i$ for some $i \in I$.

iii) $\{A_i\}_{i\in I}$ is pairwise disjoint: ✓

If $i,j \in I$ and $\underset{=\overline{x}_1}{A_i} \cap \underset{=\overline{x}_2}{A_j} \neq \emptyset$ then $A_i = A_j$. (fact 3)

Therefore, $\{A_i\}_{i\in I}$ is a partition of S.